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LOCALLY STRONGLY PRIMITIVE SEMIGROUPS OF NONNEGATIVE MATRICES

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The class of locally strongly primitive semigroups of nonnegative matrices is introduced. It is shown that, by a certain permutation similarity, all the matrices of a semigroup of the class considered can be brought to a block monomial form; moreover, any matrix product of sufficient length has positive nonzero blocks only. This shows that the following known property of an imprimitive nonnegative matrix in Frobenius form is inherited: If such a matrix is raised to a sufficiently high power, then all its nonzero blocks are positive. A combinatorial criterion of the locally strong primitivity of a semigroup of nonnegative matrices is found. Bibliography: 6 *titles.*

Introduction

By the known Frobenius theorem, an irreducible nonnegative matrix is either primitive (and then a certain power of the matrix is positive) or it is permutationally similar to a matrix of the form

$$A = \begin{pmatrix} 0 & A_{12} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & A_{r-1,r} \\ A_{r1} & 0 & \dots & 0 \end{pmatrix}, \tag{1}$$

and the diagonal blocks of the block diagonal matrix $A^r = \operatorname{diag}(A_{11}^{(r)}, A_{22}^{(r)}, \dots, A_{rr}^{(r)})$ are primitive (see, e.g., [1, p. 60]). The number $r = r(A)$ in (1) is called the index of imprimitivity of the irreducible matrix A. For a primitive matrix, the index of imprimitivity is equal to unity.

In the paper by Protasov and Voynov [2], the following generalization of the Frobenius theorem was obtained: Every irreducible semigroup of nonnegative matrices free of zero rows and columns either contains a positive matrix or, by a permutation similarity, all the matrices in the semigroup are simultaneously brought to a block monomial form (which means that in every block row and in every block column there is only one nonzero block). In addition, the transformed semigroup contains a block diagonal matrix with positive diagonal blocks.

The present paper suggests another generalization of the Frobenius theorem to the case of matrix semigroups. Frobenius' theorem implies that, starting from a certain power, all the powers of a primitive matrix A are positive. If the matrix A is imprimitive and is represented in the form (1), then all its sufficiently high powers have only positive nonzero blocks. The following generalization of the notion of a primitive matrix is known. A semigroup $\mathcal{P}$ of nonnegative matrices is said to be strongly primitive if all products of matrices from $\mathcal{P}$ with sufficiently many factors are positive [3]. In this paper, as the counterpart of an imprimitive matrix we consider a semigroup that can be brought to such a block monomial form that all sufficiently lengthy products of matrices have only positive nonzero blocks. Semigroups of nonnegative matrices possessing this property will be referred to as locally strongly primitive ones.

The paper consists of three sections. Section 1 presents a criterion of strong primitivity. Its proof implies a known estimate [3] of the exponent of strong primitivity. In Sec. 2, the notion of a completely reducible semigroup of nonnegative matrices is introduced and an analog of

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